# SEMIINFINITE COHOMOLOGY OF CONTRAGRADIENT WEYL MODULES OVER SMALL QUANTUM GROUPS

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## 1. Introduction

The present paper can be considered as a natural extension the article [Ar7]. Fix root data $(Y, X, \dots)$ of the finite type $(I, \cdot)$ and a positive integer number $\ell$. In [Ar7] we obtained a nice description for semiinfinite cohomology of the trivial module $\underline{\mathbb{C}}$ over the *small quantum group* $\mathfrak{u}_\ell$ corresponding to the root data $(Y, X, \dots)$ in terms of local cohomology of the structure sheaf on the *nilpotent cone* $\mathcal{N}$ in the corresponding semisimple Lie algebra $\mathfrak{g}$.

The geometric approach to various homological questions concerning the algebra $\mathfrak{u}_\ell$ appeared first in the pioneering paper of Ginzburg and Kumar [GK]. Let us state the main result from that paper.

**Theorem:** $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ as an associative algebra. The grading on the right hand side is provided by the action of the group $\mathbb{C}^*$ on the affine variety $\mathcal{N}$. $\square$

Moreover it was shown in [GK] that both sides of the equality carry natural structures of integrable $\mathfrak{g}$-modules and the isomorphism constructed in the paper takes the left hand side $\mathfrak{g}$-module structure to the right hand side one.

*Semiinfinite cohomology* of the trivial $\mathfrak{u}_\ell$-module was considered in [Ar1], [Ar5], [Ar6] and [Ar7]. Consider semiinfinite cohomology of a graded associative algebra $A$ (see [Ar1], [Ar2] for the definition of this cohomology theory). It is proved in [Ar2] that the algebra $\mathrm{Ext}^\bullet_A(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ acts naturally on the semiinfinite cohomology of $A$. Thus in particular for a $\mathfrak{u}_\ell$-module $M$ one can treat $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, M)$ as a quasicoherent sheaf on $\mathcal{N}$. The following statement known as the Feigin Conjecture was proved in [Ar7]. Consider the standard positive nilpotent subalgebra $\mathfrak{n}^+ \subset \mathcal{N} \subset \mathfrak{g}$.

**Theorem:** The quasicoherent sheaf on $\mathcal{N}$ provided by $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ is equal to the sheaf of local cohomology of $\mathcal{O}_\mathcal{N}$ with support on $\mathfrak{n}^+ \subset \mathcal{N}$. $\square$

Moreover note that the simply connected Lie group $G$ with the Lie algebra equal to $\mathfrak{g}$ acts naturally on $\mathcal{N}$. This action provides a structure of $\mathfrak{n}^+$-integrable $\mathfrak{g}$-modules on the local cohomology spaces $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_\mathcal{N})$. On the other hand it was shown in [Ar1] that there exists a natural $U(\mathfrak{g})$-module structure on $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$. It is proved in [Ar7] that the described $\mathfrak{g}$-module structures coincide.

Consider the *contragradient Weyl module* $\mathbb{D}W_\ell(\ell\lambda)$ over $\mathfrak{u}_\ell$ with the highest weight $\ell\lambda$. In the present paper we provide a geometric description of semiinfinite cohomology of $\mathfrak{u}_\ell$ with coefficients in $\mathbb{D}W_\ell(\ell\lambda)$. To formulate the exact statement we need some geometric notation.

Consider the flag variety $G/B$ for the group $G$ and its cotangent bundle $T^*(G/B)$. Below we denote these varieties by $\mathcal{B}$ and $\widetilde{\mathcal{N}}$ respectively.The natural projection

$\tilde{\mathcal{N}} \longrightarrow \mathcal{B}$ is denoted by $p$. The moment map for the symplectic $G$-action on $\tilde{\mathcal{N}}$ provides the *Springer-Grothendieck resolution* of singularities of the nilpotent cone $\mu: \tilde{\mathcal{N}} \longrightarrow \mathcal{N}$. Consider the linear bundle $\mathcal{L}(\lambda)$ on $\mathcal{B}$. Note that by [GK] we have

$$\mathrm{Ext}^{\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W_\ell(\ell\lambda)) = H^0(\tilde{\mathcal{N}}, p^*\mathcal{L}(\lambda)).$$

We call the following statement the generalized Feigin Conjecture. It is the main result of the paper.

**Conjecture:** The $H^0(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) = \mathrm{Ext}^{\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$-module $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W_\ell(\ell\lambda))$ is isomorphic to $H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\tilde{\mathcal{N}}, p^*\mathcal{L}(\lambda))$. $\square$

1.1. Let us describe briefly the structure of the paper. In the second section we recall in more detail results concerning both the usual and the semiinfinite cohomology of small quantum groups mentioned above. Usung Ginsburg and Kumar's description of the ordinary cohomology of $\mathfrak{u}_\ell$ with coefficients in $\mathbb{D}W_\ell(\ell\lambda)$ we construct a morphism of $H^0(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$-modules

$$\sigma: \ H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\tilde{\mathcal{N}}, p^*\mathcal{L}(\lambda)) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W_\ell(\ell\lambda)).$$

In the third section we construct a certain specialization of the quantum BGG resolution into the root of unity. We call the obtained complex *the contragradient quasi-BGG complex* and denote it by $\mathbb{D}B^\bullet_\ell(\mu)$. This complex consists of direct sums of *contragradient quasi-Verma modules* and its zero cohomology module equals $\mathbb{D}W_\ell(\mu)$. A quasi-Verma module $\mathbb{D}M^w_\ell(w\cdot\mu)$ provides a certain specialization for the family of the usual contragradient quantum Verma modules $\mathbb{D}M_\xi(w\cdot)$ defined a priori at generic values of the quantizing parameter $\xi$ into the root of unity. However it turns out that contragradient quasi-Verma modules *differ from the usual contragradient Verma modules for the algebra* $\mathbf{U}_\ell$. We show that for a prime nubmber $\ell$ the contragradient quasi-BGG complex is in fact quasiisomorphic to $\mathbb{D}W_\ell(\mu)$. It is natural to conjecture that the statement remains true for all but finitely many roots of unity but our considerations do not provide the proof in the general case.

We perform the whole construction of the quasi-BGG complex over the ring $\mathcal{A} := \mathbb{Z}[v, v^{-1},]$ and obtain a complex of $\mathbf{U}_\mathcal{A}$-modules $\mathbb{D}B^\bullet_\mathcal{A}(\mu)$ for every integral dominant weight $\mu$. Here $\mathbf{U}_\mathcal{A}$ denotes the Lusztig version of the quantum group with quantum divided powers. Next we recall the result Lusztig stating that the specialization of the algebra $\mathbf{U}_\mathcal{A}$ under the base change from $\mathcal{A}$ to $\overline{\mathbb{F}}_\ell$ for $\ell$ simple coincides up to a central extension with the specialization of the Kostant integral form for the usual universal enveloping algebra $U_\mathbb{Z}(\mathfrak{g})$ to the same field. This way we obtain a geometric interpretation of the complex $\mathbb{D}B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu)$ in terms of local cohomology of the flag variety $\mathcal{B}_{\overline{\mathbb{F}}_\ell}$ in characteristic $\ell$ as follows.

Consider the stratification of $\mathcal{B}_{\overline{\mathbb{F}}_\ell}$ by Schubert cells. For a dominant weight $\mu$ Kempf showed that the global Cousin complex $C^\bullet_{\overline{\mathbb{F}}_\ell}(\mu)$,

$$C^k_{\overline{\mathbb{F}}_\ell}(\mu) = \bigoplus_{w\in W, \ell t(w)=k} H^k_{S_{w,\overline{\mathbb{F}}_\ell}}(\mathcal{B}_{\overline{\mathbb{F}}_\ell}, \mathcal{L}(\mu)),$$

provides a resolution of the contragradient Weyl module

$$\mathbb{D}W_{\overline{\mathbb{F}}_\ell}(\mu) := H^0(\mathcal{B}_{\overline{\mathbb{F}}_\ell}, \mathcal{L}(\mu)).$$

Note also that the algebra $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$ maps naturally to the algebra of global twisted differential operators on $\mathcal{B}_{\overline{\mathbb{F}}_\ell}$ with the twist $\mathcal{L}(\mu)$ and thus it acts naturally on the above global Cousin complex. We claim that the obtained complex is isomorphic to the specialization of the quasi-BGG complex $\mathbb{D}B_{\overline{\mathbb{F}}_\ell}(\mu) := \mathbb{D}B^\bullet_{\mathcal{A}}(\mu)\otimes_{\mathcal{A}}\overline{\mathbb{F}}_\ell$. Denote the quotient ring of $\mathcal{A}$ by the ideal generated by the $\ell$-th cyclotomic polynomial by $\mathcal{A}'_\ell$

It follows that $\mathbb{D}B_\ell(\mu) = \mathbb{D}B_{\mathcal{A}'_\ell}(\mu)\otimes\mathbb{C}$ provides a resolution of $\mathbb{D}W_\ell(\mu)$ since it can be viewed as a specialization of the complex $\mathbb{D}B_{\mathcal{A}'_\ell}(\mu)$ to the generic point of $\operatorname{Spec}\mathcal{A}'_\ell$, on the other hand by the above considerations the specialization of the latter complex into Spec $\overline{\mathbb{F}}_\ell$ is exact.

Using the last result we prove the generalized Feigin conjecture. Namely we show that

$$\operatorname{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}M^w_\ell(w\cdot\ell\lambda)) = H^{\sharp(R^+)}_{T^*_{S_w}(\mathcal{B})}(\widetilde{\mathcal{N}}, \pi^*\mathcal{L}(\lambda))$$

as a module over both $U(\mathfrak{g})$ and $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}) = H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$. Here $T^*_{S_w}(\mathcal{B})$ denotes the conormal bundle to the Schubert cell $S_w$. This statement allows us to calculate semiinfinite cohomology of $\mathfrak{u}_\ell$ with coefficients in the contragradient quasi-BGG complex:

$$\operatorname{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}B^\bullet_\ell(\ell\lambda)) = H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \pi^*\mathcal{L}(\lambda)).$$

In the fourth section we formulate some further conjectures concerning possible origin of the quasi-BGG complex. A similar complex expressing a Weyl module in terms of Verma modules exists in the category $\mathcal{O}$ for the corresponding affine Lie algebra at a negative rational level. We hope that some extension of the Kazhdan-Lusztig functor takes the latter complex to our quasi-BGG complex over $\mathbf{U}_\ell$.

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## 2. Small quantum groups.

We start with recalling notations and the main results from [Ar7].

2.1. **Quantum groups at roots of unity.** Fix a *Cartan datum* $(I,\cdot)$ of the finite type and a *simply connected root datum* $(Y, X, \dots)$ of the type $(I,\cdot)$. Thus we have $Y = \mathbb{Z}[I]$, $X = \operatorname{Hom}(Y,\mathbb{Z})$, and the pairing $\langle\ ,\ \rangle:\ Y\times X \longrightarrow \mathbb{Z}$ coincides with the natural one (see [L3], I 1.1, I 2.2). In particular the data contain canonical embeddings $I \hookrightarrow Y, i\mapsto i$ and $I\hookrightarrow X, i\mapsto i' : \langle i', j\rangle := 2i\cdot j/i\cdot i$. The latter map is naturally extended to an embedding $Y\subset X$. Denote by ht the linear function on $X$ defined on elements $i', i\in I$, by $\operatorname{ht}(i') = 1$ and extended to the whole $X$ by linearity. The root system (resp. the positive root system) corresponding to the data $(Y, X, \dots)$ is denoted by $R$ (resp. by $R^+$), below $W$ denotes the Weyl group of $R$.

Like in [Ar7], Section 2, we denote the *Drinfeld-Jimbo quantum group* defined over the field $\mathbb{Q}(v)$ of rational functions in $v$ (resp. the *Lusztig version of the quantum*

*group* defined over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$) by $\mathbf{U}$ (resp. by $\mathbf{U}_{\mathcal{A}}$). Fix an odd number $\ell$ satisfying the conditions from [GK] and a primitive $\ell$-th root of unity $\xi$. Define a $\mathbb{C}$-algebra $\widetilde{\mathbf{U}}_\ell := \mathbf{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$, where $v$ acts on $\mathbb{C}$ by multiplication by $\xi$. It is known that the elements $K_i^\ell, i \in I$, are central in $\widetilde{\mathbf{U}}_\ell$. Set $\mathbf{U}_\ell := \widetilde{\mathbf{U}}_\ell/(K_i^\ell - 1, i \in I)$. The algebra $\mathbf{U}_\ell$ is generated by the elements $E_i$, $E_i^{(\ell)}$, $F_i$, $F_i^{(\ell)}$, $K_i^{\pm 1}$, $i \in I$. Here $E_i^{(\ell)}$ (resp. $F_i^{(\ell)}$) denotes the *$\ell$-th quantum divided power* of the element $E_i$ (resp. $F_i$) specialized at the root of unity $\xi$.

2.1.1. *Quantum groups in positive charavteristic.* Like in [L1] consider the specialization of $\mathbf{U}_{\mathcal{A}}$ in characteristic $\ell$. Namely let $\mathcal{A}'_\ell$ be the quotient of $\mathcal{A}$ by the ideal generated by the $\ell$-th cyclotomic polynomial. T hen $\mathcal{A}'_\ell/(v-1)$ is isomorphic to the finite field $\mathbb{F}_\ell$. Thus the algebraic closture $\overline{\mathbb{F}}_\ell$ becomes a $\mathcal{A}$-algebra. We set $\mathbf{U}_{\overline{\mathbb{F}}_\ell} := \mathbf{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathbb{F}}_\ell$. Ir is known that the elements $K_i$, $i \in I$, are central in $\mathbf{U}_{\overline{\mathbb{F}}_\ell}$ and the algebra $\mathbf{U}_{\overline{\mathbb{F}}_\ell}/(K_i, i \in I)$ is isomorphic to $U_{\mathbb{Z}}(\mathfrak{g}) \otimes \overline{\mathbb{F}}_\ell$, where $U_{\mathbb{Z}}(\mathfrak{g})$ denotes the Kostant integral form for the unicersal enveloping algebra of $\mathfrak{g}$.

2.1.2. Following Lusztig we define the *small quantum group* $\mathfrak{u}_\ell$ at the $\ell$-th root of unity $\xi$ as the subalgebra in $\mathbf{U}_\ell$ generated by all $E_i, F_i, K_i^{\pm 1}, i \in I$. Denote the subalgebra in $\mathfrak{u}_\ell$ generated by $E_i, i \in I$ (resp. $F_i, i \in I$, resp. $K_i, i \in I$), by $\mathfrak{u}_\ell^+$ (resp. $\mathfrak{u}_\ell^-$, resp. $\mathfrak{u}_\ell^0$). The subalgebra $\mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^0$ (resp. $\mathfrak{u}_\ell^0 \otimes \mathfrak{u}_\ell^+$) in $\mathfrak{u}_\ell$ is denoted by $\mathfrak{b}_\ell^-$ (resp. by $\mathfrak{b}_\ell^+$).

Recall that a finite dimensional algebra $A$ is called *Frobenius* if the left $A$-modules $A$ and $A^* := \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ are isomorphic.

2.1.3. **Lemma:** (see [Ar1], Lemma 2.4.5) The algebras $\mathfrak{u}_\ell^+$ and $\mathfrak{u}_\ell^-$ are Frobenius. □

Note that the algebra $\mathfrak{u}_\ell$ is graded naturally by the abelian group $X$. Using the function ht we obtain a $\mathbb{Z}$-grading on $\mathfrak{u}_\ell$ from this $X$-grading. In particular the subalgebra $\mathfrak{u}_\ell^+$ (resp. $\mathfrak{u}_\ell^-$) is graded by $\mathbb{Z}_{\geq 0}$ (resp. by $\mathbb{Z}_{\leq 0}$).

Below we present several well known facts about the algebra $\mathfrak{u}_\ell$ to be used later. Recall that an augmented subalgebra $B \subset A$ with the augmentation ideal $\overline{B} \subset B$ is called *normal* if $A\overline{B} = \overline{B}A$. If so, the space $A/A\overline{B}$ becomes an algebra. It is denoted by $A//B$. Fix an augmentation on $\mathfrak{u}_\ell$ as follows: $E_i \mapsto 0, F_i \mapsto 0, K_i \mapsto 1$ for every $i \in I$. Set $\underline{\mathbb{C}} := \mathfrak{u}_\ell/\overline{\mathfrak{u}}_\ell$.

2.1.4. **Lemma:** (see [AJS] 1.3, [L2] Theorem 8.10)

(i) The multiplication in $\mathfrak{u}_\ell$ provides a vector space isomorphism $\mathfrak{u}_\ell = \mathfrak{u}_\ell^- \otimes \mathfrak{u}_\ell^0 \otimes \mathfrak{u}_\ell^+$; the subalgebra $\mathfrak{u}_\ell^0$ is isomorphic to the group algebra of the group $(\mathbb{Z}/\ell\mathbb{Z})^{\sharp(I)}$.

(ii) The subalgebra $\mathfrak{u}_\ell \subset \mathbf{U}_\ell$ is normal and we have $\mathbf{U}_\ell//\mathfrak{u}_\ell = U(\mathfrak{g})$. □

Denote the category of $X$-graded finite dimensional left $\mathfrak{u}_\ell$-modules $M = \bigoplus\limits_{\lambda \in X} M_\lambda$ such that $K_i$ acts on $M_\lambda$ by multiplication by the scalar $\xi^{\langle i, \lambda \rangle}$ and $E_i: M_\lambda \longrightarrow M_{\lambda + i'}$, $F_i: M_\lambda \longrightarrow M_{\lambda - i'}$ for all $i \in I$, with morphisms preserving $X$-gradings, by $\mathfrak{u}_\ell$-mod. For $M, N \in \mathfrak{u}_\ell$-mod and $\lambda \in \ell \cdot X$ we define the shifted module $M\langle\lambda\rangle \in \mathfrak{u}_\ell$-mod : $M\langle\lambda\rangle_\mu := M_{\lambda+\mu}$ and set $\mathrm{Hom}_{\mathfrak{u}_\ell}(M, N) := \bigoplus\limits_{\lambda \in \ell \cdot X} \mathrm{Hom}_{\mathfrak{u}_\ell\text{-mod}}(M\langle\lambda\rangle, N)$. Evidently the spaces $\mathrm{Hom}_{\mathfrak{u}_\ell}(\cdot, \cdot)$ and posess natural $\ell \cdot X$-gradings.

2.2. **Cohomology of small quantum groups.** Consider the $\ell \cdot X \times \mathbb{Z}$-graded algebra $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$. Note that by Shapiro lemma and Lemma 2.1.4 (ii) the Lie algebra $\mathfrak{g}$ acts naturally on the Ext algebra and the multiplication in the algebra satisfies Lebnitz rule with respect to the $\mathfrak{g}$-action. In [GK] Ginzburg and Kumar obtained a nice description of the multiplication structure as well as the $\mathfrak{g}$-module structure on $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ as follows.

2.2.1. *Functions on the nilpotent cone.* Let $G$ be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Then $G$ acts on $\mathfrak{g}$ by adjunction. The action preserves the set of nilpotent elements $\mathcal{N} \subset \mathfrak{g}$ called the *nilpotent cone* of $\mathfrak{g}$. The action is algebraic, thus it provides a morphism of $\mathfrak{g}$ into the Lie algebra of algebraic vector fields on the nilpotent cone $\mathcal{V}ect(\mathcal{N})$. The latter algebra acts on the algebraic functions $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$. The action is $G$-integrable. Note also that the natural action of the group $\mathbb{C}^*$ provides a grading on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ preserved by the $G$-action.

2.2.2. **Theorem:** (see [GK]) The algebra and $\mathfrak{g}$-module structures on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ and on $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ coincide. The homological grading on the latter algebra corresponds to the grading on the former one provided by the $\mathbb{C}^*$-action. The $X$-grading on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ provided by the weight decomposition with respect to the action of the Cartan subalgebra in $\mathfrak{g}$ corresponds to the natural $\ell \cdot X$-grading on the space $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$. $\square$

2.3. **Semiinfinite cohomology of small quantum groups.** Here we present the definition of semiinfinite cohomology of a finite dimensional associative algebra $\mathfrak{u}$. The setup for the definition includes a nonnegatively (resp. nonpositively) graded subalgebras $\mathfrak{u}^+$ and $\mathfrak{u}^-$ in $\mathfrak{u}$ such that the multiplication in $\mathfrak{u}$ provides a vector space isomorphism $\mathfrak{u}^- \otimes \mathfrak{u}^+ \widetilde{\longrightarrow} \mathfrak{u}$. Below we suppose that the algebra $\mathfrak{u}^+$ is Frobenius. Note still that the general definition of associative algebra semiinfinite cohomology given in [Ar1] requires neither this restriction nor the assumption that $\dim \mathfrak{u} < \infty$.

Consider first the *semiregular* $\mathfrak{u}$-bimodule $S_\mathfrak{u}^{\mathfrak{u}^+} = \mathfrak{u} \otimes_{\mathfrak{u}^+} \mathfrak{u}^{+*}$, with the right $\mathfrak{u}$-module structure provided by the isomorphism of left $\mathfrak{u}^+$-modules $\mathfrak{u}^+ \cong \mathfrak{u}^{+*}$. Note that $S_\mathfrak{u}^{\mathfrak{u}^+}$ is free over the algebra $\mathfrak{u}$ both as a right and as a left module.

For a complex of graded $\mathfrak{u}$-modules $M^\bullet = \bigoplus_{p,q \in \mathbb{Z}} M_p^q, \ d: \ M_p^q \longrightarrow M_p^{q+1}$ we define the support of $M^\bullet$ by $\mathrm{supp}\, M^\bullet := \{(p,q) \in \mathbb{Z}^2 | M_p^q \neq 0\}$. We say that a complex $M^\bullet$ is *concave* (resp. *convex*) if there exist $s_1, s_2 \in \mathbb{N}, t_1, t_2 \in \mathbb{Z}$ such that $\mathrm{supp}\, M^\bullet \subset \{(p,q) \in \mathbb{Z}^2 | s_1 q + p \leq t_1, s_2 q - p \leq t_2\}$ (resp. $\mathrm{supp}\, M^\bullet \subset \{(p,q) \in \mathbb{Z}^2 | s_1 q + p \geq t_1, s_2 q - p \geq t_2\}$).

Let $M^\bullet, N^\bullet \in \mathcal{C}om(\mathfrak{u}\text{-mod})$. Suppose that $M^\bullet$ is convex and $N^\bullet$ is concave. Choose a convex (resp. concave) complex $R^\bullet_\uparrow(M^\bullet)$ (resp. $R^\bullet_\downarrow(N^\bullet)$) in $\mathcal{C}om(\mathfrak{u}\text{-mod})$ quasi-isomorphic to $M^\bullet$ (resp. $N^\bullet$) and consisting of $\mathfrak{u}^+$-projective (resp. $\mathfrak{u}^-$-projective) modules.

2.3.1. **Definition:** We set

$$\mathrm{Ext}_\mathfrak{u}^{\frac{\infty}{2}+\bullet}(M^\bullet, N^\bullet) := H^\bullet(\mathrm{Hom}^\bullet_\mathfrak{u}(R^\bullet_\uparrow(M^\bullet), S_\mathfrak{u}^{\mathfrak{u}^+} \otimes_\mathfrak{u} R^\bullet_\downarrow(N^\bullet))).$$

2.3.2. **Lemma:** (see. [Ar1] Lemma 3.4.2, Theorem 5) The spaces $\mathrm{Ext}_{\mathfrak{u}}^{\frac{\infty}{2}+\bullet}(M^\bullet, N^\bullet)$ do not depend on the choice of resolutions and define functors

$$\mathrm{Ext}_{\mathfrak{u}}^{\frac{\infty}{2}+k}(\cdot,\cdot):\ \mathfrak{u}\text{-mod}\times\mathfrak{u}\text{-mod}\longrightarrow \mathcal{V}ect,\ k\in\mathbb{Z}.\quad\square$$

Below we consider semiinfinite cohomology of algebras $\mathfrak{u}_\ell$, $\mathfrak{b}_\ell^+$, $\mathfrak{b}_\ell^-$ etc. with coefficients in $X$-graded modules. The $\mathbb{Z}$-grading on such a module is obtained from the $X$-grading using the function $\mathrm{ht}:\ X\longrightarrow\mathbb{Z}$.

Evidently the spaces $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(M^\bullet, N^\bullet)$ posess natural $\ell\cdot X$-gradings. The following statement is a direct consequence of Lemma 2.1.4(ii).

2.3.3. **Lemma:** Let $M,N\in\mathfrak{u}_\ell$-mod be restrictions of some $\mathbf{U}_\ell$-modules. Then the spaces $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(M,N)$ have natural structures of $\mathfrak{g}$-modules, and the $\ell\cdot X$-gradings on them coincide with the $X$-gradings provided by the weight decompositions of the modules with respect to the standard Cartan subalgebra in $\mathfrak{g}$. $\square$

2.3.4. *Semiinfinite cohomology of the trivial $\mathfrak{u}_\ell$-module.* Note that for a $\mathbf{U}_\ell$-module $M$ the space $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, M)$ has a natural structure of a module over the algebra $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ and this structure is equivariant with respect to the action of the Lie algebra $\mathfrak{g}$.

Let us recall the geometric description of semiinfinite cohomology of the trivial $\mathfrak{u}_\ell$-module obtained in [Ar7]. Consider the standard positive nilpotent subalgebra $\mathfrak{n}^+\subset\mathfrak{g}$ as a Zarisski closed subset in $\mathcal{N}$. Consider the $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_\mathcal{N})$ of local cohomology with supports in $\mathfrak{n}^+\subset\mathcal{N}$ for the structure sheaf $\mathcal{O}_\mathcal{N}$. Recall also that the space $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_\mathcal{N})$ has a natural $\mathfrak{g}$-module structure defined as follows. First the Lie algebra $\mathcal{V}ect(\mathcal{N})$ acts naturally on the local cohomology space. Now the adjoint action of $G$ on $\mathcal{N}$ defines a Lie algebra inclusion $\mathfrak{g}\subset\mathcal{V}ect(\mathcal{N})$. Note that the $H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$-module $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_\mathcal{N})$ is evidently $\mathfrak{g}$-equivariant with respect to the defined actions of $\mathfrak{g}$ on the algebra and the module.

Note also that the subset $\mathfrak{n}^+\subset\mathcal{N}$ is $\mathbb{C}^*$-stable, thus the space $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_\mathcal{N})$ is naturally graded by the $\mathbb{C}^*$-action.

The following statement sums up the main results from [Ar7].

**Theorem:**

(i) Semiinfinite cohomology of the trivial $\mathfrak{u}_\ell$-module vanishes in even degrees.

(ii) The space $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+2k+1}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ is isomorphic to the homogeneous component in $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_\mathcal{N})$ of the weight $k$.

(iii) $H_{\mathfrak{n}^+}^{\sharp(R^+)}(\mathcal{N},\mathcal{O}_\mathcal{N})$ is isomorphic to $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ both as a module over $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})=H^0(\mathcal{N},\mathcal{O}_\mathcal{N})$ and as a $\mathfrak{g}$-module. $\square$

2.3.5. *Springer-Grothendieck resolution of the nilpotent cone.* We provide another description of the space $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\underline{\mathbb{C}})$ in terms of local cohomology as follows.

Choose a maximal torus $H$ in the simply connected Lie group $G$ with the Lie algebra $\mathfrak{g}$. The choice provides the root decomposition of $\mathfrak{g}$ and in particular its triangular decomposition $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$. Consider the Borel subgroup $B\subset G$ with the Lie algebra $\mathfrak{b}^+=\mathfrak{h}\oplus\mathfrak{n}^+$ and the flag variety $\mathcal{B}$. The group $G$ acts on $\mathcal{B}$ by left translations and the restriction of this action to $B$ is known to have finitely

many orbits. These orbits are isomorphic to affine spaces and called *the Schubert cells.* The Bruhat decomposition of $G$ shows that the Schubert cells are enumerated by the Weyl group. Denote the orbit corresponding to the element $w \in W$ by $S_w$.

It is well known that the cotangent bundle $T^*(\mathcal{B}) =: \widetilde{\mathcal{N}}$ has a nice realization $\widetilde{\mathcal{N}} = \{(B_x, n) | n \in \mathrm{Lie}(B_x)\}$, where $B_x$ denotes some Borel subgroup in $G$ and $n$ is a nilpotent element in the Lie algebra $\mathrm{Lie}(B_x)$. The map

$$\mu : \ \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}, \ (B_x, n) \mapsto n,$$

is known to be a resolution of singularities of $\mathcal{N}$ called *the Springer-Grothendieck resolution.* Note that the map $\mu$ is $G$-equivariant.

Recall the following statement from [Ar1].

2.3.6. **Proposition:** (see e. g. [CG], 3.1.36)

(i) $\mu^{-1}(\mathfrak{n}^+) = \bigsqcup_{w \in W} T^*_{S_w}(\mathcal{B})$, where $T^*_{S_w}(\mathcal{B})$ denotes the conormal bundle to $S_w$ in $\mathcal{B}$.

(ii) $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}) = H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ and the higher cohomology spaces of the structure sheaf on $\widetilde{\mathcal{N}}$ vanish.

(iii) $H^{\sharp(R^+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_{\mathcal{N}}) \widetilde{\longrightarrow} H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})$ both as a $\mathfrak{g}$-module and as a $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-module. $\square$

**Corollary:** $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ is isomorphic to $H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})$ both as a module over $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})$ and as a $\mathfrak{g}$-module. $\square$

2.4. **Semiinfinite cohomology of contragradient Weyl modules.** Our main goal now is to find a nice geometric interpretation of semiinfinite cohomology of some remarkable $\mathfrak{u}_\ell$-modules that would generalize the results of stated in the previous section.

Fix the natural triangular decompositions of the algebra $\mathbf{U}_{\mathcal{A}}$ (resp. $\mathbf{U}_\ell$) as follows: $\mathbf{U}_{\mathcal{A}} = \mathbf{U}^-_{\mathcal{A}} \otimes \mathbf{U}^0_{\mathcal{A}} \otimes \mathbf{U}^+_{\mathcal{A}}$ (resp. $\mathbf{U}_\ell = \mathbf{U}^-_\ell \otimes \mathbf{U}^0_\ell \otimes \mathbf{U}^+_\ell$), where the positive (resp. negative) subalgebras are generated by the quantum divided powers of the positive (resp. negative) root generators in the corresponding algebra. We call the subalgebra $\mathbf{U}^+_{\mathcal{A}} \otimes \mathbf{U}^0_{\mathcal{A}}$ (resp. $\mathbf{U}^+_\ell \otimes \mathbf{U}^0_\ell$) the *positive quantum Borel subalgebra* in $\mathbf{U}_{\mathcal{A}}$ (resp. in $\mathbf{U}_\ell$) and denote it by $\mathbf{B}^+_{\mathcal{A}}$ (resp. by $\mathbf{B}^+_\ell$). The negative Borel subalgebras $\mathbf{B}^-_{\mathcal{A}}$ and $\mathbf{B}^-_\ell$ are defined in a similar way.

2.5. Fix a dominant integral weight $\lambda \in X$. Consider the module over the "big" quantum group $\mathbf{U}_{\mathcal{A}}$ given by

$$\mathbb{D}W_{\mathcal{A}}(\lambda) := \left(\mathrm{Coind}^{\mathbf{U}_{\mathcal{A}}}_{\mathbf{B}^-_{\mathcal{A}}} \mathbb{C}(\lambda)\right)^{\mathrm{fin}} \text{ (resp. by } W_{\mathcal{A}}(\lambda) := \left(\mathrm{Ind}^{\mathbf{U}_\ell}_{\mathbf{B}^+_\ell} \mathbb{C}(\lambda)\right)_{\mathrm{fin}}).$$

Here $(*)^{\mathrm{fin}}$ (resp. $(*)_{\mathrm{fin}}$) denotes the maximal finite dimensional submodule (resp. quotient module) in $(*)$. The module $\mathbb{D}W_{\mathcal{A}}(\lambda)$ (resp. $W_{\mathcal{A}}(\lambda)$) is called *the contragradient Weyl module* (resp. *the Weyl module*) over $\mathbf{U}_{\mathcal{A}}$ with the highest weight $\lambda$. We denote by $\mathbb{D}W_\ell(\lambda)$ (resp. by $W_\ell(\lambda)$) the specializations of the corresponding modules into the chosen $\ell$-th root of unity.

It is known that at the generic value $\xi$ of the quantizing parameter $v$ both modules $W_{\mathcal{A}}(\lambda)$ and $\mathbb{D}W_{\mathcal{A}}(\lambda)$ specialize into the finite dimensional simple module $L(\lambda)$ over

the quantum group $\mathbf{U}_\xi := \mathbf{U}_{\mathcal{A}}\otimes_{\mathcal{A}}\mathbb{C}$. In particular we have

$$\mathrm{ch}(W_{\mathcal{A}}(\lambda)) = \mathrm{ch}(\mathbb{D}W_{\mathcal{A}}(\lambda)) = \sum_{w\in W} \frac{e^{w\cdot\lambda}}{\prod_{\alpha\in R^+}(1-e^{\alpha})},$$

just like in the Weyl character formula in the semisimple Lie algebra case. Note also that $W_{\mathcal{A}}(0) = \mathbb{D}W_{\mathcal{A}}(0) = \underline{\mathbb{C}}$.

Below we consider semiinfinite cohomology of the algebra $\mathfrak{u}_\ell$ with coefficients in the contragradient Weyl module with a $\ell$-divisible highest weight $\ell\lambda$. Ou considerations were motivated by the following result of Ginzburg and Kumar (see [GK]).

Let $p$ denote the projection $\widetilde{\mathcal{N}} \longrightarrow \mathcal{B}$. Consider the linear bundle $\mathcal{L}(\lambda)$ on $\mathcal{B}$ with the first Chern class equal to $\lambda \in X = H^2(\mathcal{B},\mathbb{Z})$.

2.5.1. **Theorem:**

(i) $\mathrm{Ext}^{\mathrm{odd}}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W_\ell(\ell\lambda)) = 0$;

(ii) $\mathrm{Ext}^{2\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W_\ell(\ell\lambda)) = H^0(\widetilde{\mathcal{N}}, p^*\mathcal{L}(\lambda))$ as a $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-module. $\square$

The following conjecture provides a natural semiinfinite analogue for Theorem 2.5.1.

2.5.2. **Conjecture:** $\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W_\ell(\ell\lambda)) = H^0_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, p^*\mathcal{L}(\lambda))$ as a $H^0(\mathcal{N},\mathcal{O}_{\mathcal{N}})$-module. The homological grading on the left hand side of the equality corresponds to the grading by the natural action of the group $\mathbb{C}^*$ on the right hand side. $\square$

2.5.3. **Corollary:**

$$\mathrm{ch}\left(\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W_\ell(\ell\lambda)), t\right)$$

$$= \frac{t^{-\sharp(R^+)}}{\prod_{\alpha\in R^+}(1-e^{-\ell\alpha})} \sum_{w\in W} \frac{e^{w(\ell\lambda)}t^{2l(w)}}{\prod_{\alpha\in R^+, w(\alpha)\in R^+}(1-t^2e^{-\ell\alpha})\prod_{\alpha\in R^+, w(\alpha)\in R^-}(1-t^{-2}e^{-\ell\alpha})}. \quad \square$$

Below we present the main steps for the proof of the conjecture. Details of the proof will be given in the fourthcoming paper [Ar8].

2.6. **Local cohomology with coefficients in $p^*\mathcal{L}(\lambda)$.** For a quasicoherent sheaf $\mathcal{M}$ on $\widetilde{\mathcal{N}}$ consider the natural pairing $H^i_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}) \times H^0(\widetilde{\mathcal{N}}, \mathcal{M}) \longrightarrow H^i_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \mathcal{M})$. Evidently it is equivariant with respect to the $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})$-action. Thus we obtain a $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})$-module morphism

$$s:\ H^i_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})\otimes_{H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})} H^0(\widetilde{\mathcal{N}}, \mathcal{M}) \longrightarrow H^i_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \mathcal{M}).$$

2.6.1. **Proposition:** For $\mathcal{M} = p^*\mathcal{L}(\lambda)$ the map $s$ is an isomorphism. $\square$

2.6.2. *Similar construction for semiinfinite cohomology.* We will need some more homological algebra. Fix a graded algebra $A$ with a subalgebra $B \subset A$. Recall that in [V] and [Ar1] the notion of a complex of graded $A$-modules K-semijective with respect to the subalgebra $B$ was developed. The following statement gives an analogue of the standard technique of projective resolutions in the semiinfinite case.

2.6.3. **Theorem:** (see [Ar1], Appendix B) Let $SS^\bullet_{\mathfrak{u}_\ell^+}(*)$ (resp. $SS^\bullet_{\mathfrak{u}_\ell^-}(*)$) denote a K-semijective concave (resp. convex) resolution of the $\mathfrak{u}_\ell$-module $(*)$ with respect to the subalgebra $\mathfrak{u}_\ell^+$ (resp. $\mathfrak{u}_\ell^-$). Then for a finite dimensional graded $\mathfrak{u}_\ell$-module $M$ we have

(i) $H^\bullet(\mathrm{Hom}^\bullet_{\mathfrak{u}_\ell}(SS^\bullet_{\mathfrak{u}_\ell^-}(\underline{\mathbb{C}}), SS^\bullet_{\mathfrak{u}_\ell^-}(M)) = \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, M)$;

(ii) $H^\bullet(\mathrm{Hom}^\bullet_{\mathfrak{u}_\ell}(SS^\bullet_{\mathfrak{u}_\ell^+}(\underline{\mathbb{C}}), SS^\bullet_{\mathfrak{u}_\ell^-}(\underline{\mathbb{C}})) = \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, M)$. □

2.6.4. **Corollary:** The composition of morphisms provides a natural pairing

$$\mathrm{Ext}^{\frac{\infty}{2}+i}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \times \mathrm{Ext}^j_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, M) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+i+j}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, M). \quad \square$$

In particular we obtain a $\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})$-module map

$$\mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \otimes_{\mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \underline{\mathbb{C}})} \mathrm{Ext}^\bullet_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, M) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, M).$$

Combining this construction with the previous considerations we obtain the following statement.

2.6.5. **Proposition:** There exists a natural $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-module morphism

$$\sigma: \ H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, p^*\mathcal{L}(\lambda)) \longrightarrow \mathrm{Ext}^{\frac{\infty}{2}+\bullet}_{\mathfrak{u}_\ell}(\underline{\mathbb{C}}, \mathbb{D}W(\ell\lambda)).$$

**Proof.** Follows from Proposition 2.6.1. □

Below we show that the morphism $\sigma$ is an isomorphism. The main tool for the demonstration of this fact is the *quasi-BGG complex* providing a specialization of the classical BGG resolution for a finite dimensional simple $\mathbf{U}$-module $L(\ell\lambda)$ into the root of unity $\xi$. This complex constructed below consists of direct sums of $\mathbf{U}_\ell$-modules called *the quasi-Verma modules*. On the other hand we show that semiinfinite cohomology with coefficients in quasi-Verma modules has a nice geometrical interpretation.

## 3. The construction of the quasi-BGG complex.

Recall that the usual Bernshtein-Gelfand-Gelfand resolution of a finite dimensional representatiln $L(\lambda)$ of the simple Lie algebra $\mathfrak{g}$ has a nice geometric interpretation as follows. First by Borel-Weyl-Bott theorem we have $L(\lambda) = H^)(\mathcal{B}, \mathcal{L}(\lambda))$. Next consider the sratification $\{S_w, w \in W\}$ of the flag variety by the Schubert cells. Kempf showed that the contragradient BGG-resolution of $L(\lambda)$ coincides with the global Cousin complex $\mathcal{C}^\bullet(\lambda)$:

$$\mathcal{C}^k(\lambda) = \bigoplus_{w \in W, \ell t(w)=k} H^k_{S_w}(\mathcal{B}, \mathcal{L}(\lambda)).$$

In particular this construction inedtifies the local cohomology space $H^k_{S_w}(\mathcal{B}, \mathcal{L}(\lambda))$ with the contragradient Verma module $\mathbb{D}M(w \cdot \lambda)$.

3.1. **Cousin complexes in positive characteristic.** In fact the local cohomology construction due to Kempf works in a more general setting. Let $\overline{\mathbb{F}}_\ell$ be the algebraic closture of the finite field of characteristic $\ell$. It is known that both the flag variety for $\mathfrak{g}$ and its stratification by Schubert cells are well defined over $\overline{\mathbb{F}}_\ell$. We denote the corresponding objectsby $\mathcal{B}_{\overline{\mathbb{F}}_\ell}$ and $S_{w,\overline{\mathbb{F}}_\ell}$ respectively.

Let $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$ be the specialization of the Kostant integral form for the universal enveloping algebra $U_{\mathbb{Z}}(\mathfrak{g})$ into $\overline{\mathbb{F}}_\ell$. Fix a dominant weight $\lambda$. Then it is known that just like in the complex case $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$ maps naturally into the algebra of global differential operators on $\mathcal{B}_{\overline{\mathbb{F}}_\ell}$ with coefficients in the line bundle $\mathcal{L}(\lambda)$ denoted by $\mathrm{Diff}(\mathcal{L}(\lambda))$.

Consider the contragradient Weyl module over $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$ defined as follows:

$$\mathbb{D}W_{\overline{\mathbb{F}}_\ell}(\lambda) := \left(\mathrm{Coind}_{U_{\overline{\mathbb{F}}_\ell}(\mathfrak{b}^-)}^{U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})} \overline{\mathbb{F}}_\ell(\lambda)\right)^{\mathrm{fin}}.$$

The Borel-Weyl-Bott theorem remains true in prime characteristic.

**Theorem:**

(i) $H^{>0}(\mathcal{B}_{\overline{\mathbb{F}}_\ell}, \mathcal{L}(\lambda)) = 0$;

(ii) $H^0(\mathcal{B}_{\overline{\mathbb{F}}_\ell}, \mathcal{L}(\lambda)) = \mathbb{D}W(\lambda)$. □

What is even more important for us is that the Kempf construction of the global Cousin complex works over $\overline{\mathbb{F}}_\ell$ as well.

3.1.1. **Theorem:** (see [K]) For any dominant weight $\lambda$ there exists a complex $\mathcal{C}^\bullet_{\overline{\mathbb{F}}_\ell}(\lambda)$,

$$\mathcal{C}^k_{\overline{\mathbb{F}}_\ell}(\lambda) := \bigoplus_{w\in W, \ell t(w)=k} H^k_{S_{w,\overline{\mathbb{F}}_\ell}}(\mathcal{B}_{\overline{\mathbb{F}}_\ell}, \mathcal{L}(\lambda)).$$

**Remark:** The algebra $\mathrm{Diff}(\mathcal{L}(\lambda))$ acts naturally on $\mathcal{C}^\bullet_{\overline{\mathbb{F}}_\ell}(\lambda)$ and this complex becomes the one of $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$-modules. An important difference from the complex case is that the complex no longer consists of direct sums of contragradient Verma modules.

Our aim is to mimick algebraically the above construction of the global Cousin complex in the setting of the quantum group $\mathbf{U}_\ell$ rather than the algebra $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$.

3.2. **Twisted quantum parabolic subalgebras in $\mathbf{U}_\mathcal{A}$.** Recall that Lusztig has constructed an action of the *braid group* $\mathfrak{B}$ corresponding to the Cartan data $(I,\cdot)$ by automorphisms of the quantum group $\mathbf{U}_\mathcal{A}$ well defined with respect to the $X$-gradings (see [L1], Theorem 3.2). Fix a reduced expression of the maximal length element $w_0\in W$ via the simple reflection elements:

$$w_0 = s_{i_1}\dots s_{i_{\sharp(R^+)}},\ i_k\in I.$$

Then it is known that this reduced expression provides reduced expressions for all the elements $w\in W$: $w = s_{i_1^w}\dots s_{i^w_{l(w)}},\ i_k\in I$.

Consider the standard generators $\{T_i\}_{i\in I}$ in the braid group $\mathfrak{B}$. Lifting the reduced expressions for the elements $w$ from $W$ into $\mathfrak{B}$ we obtain the set of elements in the braid group of the form $T_w := T_{i_1^w}\dots T_{i^w_{l(w)}}$.

In particular we obtain the set of *twisted Borel subalgebras* $w(\mathbf{B}^+_\mathcal{A}) = T_w(\mathbf{B}^+_\mathcal{A})\subset \mathbf{U}_\mathcal{A}$. Note that $w_0(\mathbf{B}^+_\mathcal{A}) = \mathbf{B}^-_\mathcal{A} = \mathbf{U}^-_\mathcal{A}\otimes\mathbf{U}^0_\mathcal{A}$.

Fix a subset $J\subset I$ and consider the *quantum parabolic subalgebra* $\mathbf{P}_{J,\mathcal{A}}\subset\mathbf{U}_\mathcal{A}$. By definition this subalgebra in $\mathbf{U}_\mathcal{A}$ is generated over $\mathbf{U}^0_\mathcal{A}$ by the elements $E_i$, $i\in I$, $F_j$,

$j \in J$, and by their quantum divided powers. The previous construction provides the set of *twisted quantum parabolic subalgebras* $w(\mathbf{P}_{J,\mathcal{A}}) := T_w(\mathbf{P}_{J,\mathcal{A}})$ of the type $J$ with the twists $w \in W$.

Note that the triangular decomposition of the algebra $\mathbf{U}_{\mathcal{A}}$ provides the ones for the algebras $w(\mathbf{B}_{\mathcal{A}}^+)$ and $w(\mathbf{P}_{J,\mathcal{A}})$:

$$w(\mathbf{B}_{\mathcal{A}}^+) = (w(\mathbf{B}_{\mathcal{A}}^+))^- \otimes \mathbf{U}_{\mathcal{A}}^0 \otimes (w(\mathbf{B}_{\mathcal{A}}^+))^+ \text{ and } w(\mathbf{P}_{J,\mathcal{A}}) = (w(\mathbf{P}_{J,\mathcal{A}}))^- \otimes \mathbf{U}_{\mathcal{A}}^0 \otimes (w(\mathbf{P}_{J,\mathcal{A}})^+,$$

where $\left(w(\mathbf{B}_{\mathcal{A}}^+)\right)^+ = w(\mathbf{B}_{\mathcal{A}}^+) \cap \mathbf{U}_{\mathcal{A}}^-$, $(w(\mathbf{P}_{J,\mathcal{A}}))^+ = w(\mathbf{P}_{J,\mathcal{A}}) \cap \mathbf{U}_{\mathcal{A}}^-$ etc.

Choosing the root of unity $\xi$ and specializing the algebra $\mathcal{A}$ into $\mathbb{C}$, where $v$ acts on $\mathbb{C}$ by the multiplication by $\xi$, we obtain in particular the subalgebras $w(\mathbf{B}_\ell^+) \subset \mathbf{U}_\ell$, $w(\mathbf{P}_{J,\ell}) \subset \mathbf{U}_\ell$, $w(\mathfrak{b}_\ell^+) \subset \mathfrak{u}_\ell$, $w(\mathfrak{p}_{J,\ell}) \subset \mathfrak{u}_\ell$ with the induced triangular decompositions.

3.3. **Semiinfinite induction and coinduction.** From now on we will use freely the technique of associative algebra semiinfinite homology and cohomology for a graded associative algebra $A$ with two subalgebras $B, N \subset A$ equipped with a triangular decomposition $A = B \otimes N$ on the level of graded vector spaces. We will not recall the construction of these functors referring the reader to [Ar1] and [Ar2].

Let us mention only that these functors are bifunctors $\mathsf{D}(A\text{-mod}) \times \mathsf{D}(A^\sharp\text{-mod}) \longrightarrow \mathsf{D}(\mathcal{V}ect)$ where the associative algebra $A^\sharp$ is defined as follows.

Consider the semiregular $A$-module $S_A^N := A \otimes_N N^*$. It is proved in [Ar2] that under very weak conditions on the algebra $A$ the module $S_A^N$ is isomorphic to the $A$-module $\left(S_A^N\right)' := \mathrm{Hom}_B(A, B)$. Thus $\mathrm{End}_A(S_A^N) \supset N^{\mathrm{opp}}$ and $\mathrm{End}_A(S_A^N) \supset B^{\mathrm{opp}}$ as subalgebras. The algebra $A^\sharp$ is defined as the subalgebra in $\mathrm{End}_A(S_A^N)$ generated by $B^{\mathrm{opp}}$ and $N^{\mathrm{opp}}$. It is proved in [Ar2] that the algebra $A^\sharp$ has a triangular decomposition $A^\sharp = N^{\mathrm{opp}} \otimes B^{\mathrm{opp}}$ on the level of graded vector spaces. Yet for an arbitrary algebra $A$ the algebras $A^\sharp$ and $A^{\mathrm{opp}}$ do not coincide.

However the following statement shows that in the case of quantum groups that correspond to the root data $(Y, X, \dots)$ of the *finite* type $(I, \cdot)$. the equality of $A^{\mathrm{opp}}$ and $A^\sharp$ holds.

3.3.1. **Proposition:** We have

(i) $\mathbf{U}^\sharp = \mathbf{U}^{\mathrm{opp}}$, $\mathbf{U}_{\mathcal{A}}^\sharp = \mathbf{U}_{\mathcal{A}}^{\mathrm{opp}}$, $\mathbf{U}_\ell^\sharp = \mathbf{U}_\ell^{\mathrm{opp}}$;

(ii) $w(\mathbf{B}^+)^\sharp = w(\mathbf{B}^+)^{\mathrm{opp}}$, $w(\mathbf{B}_{\mathcal{A}}^+)^\sharp = w(\mathbf{B}_{\mathcal{A}}^+)^{\mathrm{opp}}$, $w(\mathbf{B}_\ell^+)^\sharp = w(\mathbf{B}_\ell^+)^{\mathrm{opp}}$, $w(\mathbf{P}_{J,\mathcal{A}})^\sharp = w(\mathbf{P}_{J,\mathcal{A}})^{\mathrm{opp}}$, $w(\mathbf{P}_{J,\ell})^\sharp = w(\mathbf{P}_{J,\ell})^{\mathrm{opp}}$.

**Proof.** The first part is proved similarly to Lemma 9.4.1 from [Ar6]. The second one follows immediately from the first one. $\square$

3.3.2. **Definition:** Let $M^\bullet$ be a convex complex of $w(\mathbf{B}_{\mathcal{A}}^+)$-modules. By definition set

$$\text{S-Ind}_{w(\mathbf{B}_{\mathcal{A}}^+)}^{\mathbf{U}_{\mathcal{A}}}(M^\bullet) := \mathrm{Tor}_{\frac{\infty}{2}+0}^{w(\mathbf{B}_{\mathcal{A}}^+)}(S_{\mathbf{U}_{\mathcal{A}}}^{\mathbf{U}_{\mathcal{A}}^+}, M^\bullet) \text{ and}$$

$$\text{S-Coind}_{w(\mathbf{B}_{\mathcal{A}}^+)}^{\mathbf{U}_{\mathcal{A}}}(M^\bullet) := \mathrm{Ext}_{w(\mathbf{B}_{\mathcal{A}}^+)}^{\frac{\infty}{2}+0}(S_{\mathbf{U}_{\mathcal{A}}}^{\mathbf{U}_{\mathcal{A}}^-}, M^\bullet).$$

The functors $\text{S-Ind}_{w(\mathbf{B}_\ell^+)}^{\mathbf{U}_\ell}(\cdot)$, $\text{S-Coind}_{w(\mathbf{B}_\ell^+)}^{\mathbf{U}_\ell}(\cdot)$, $\text{S-Ind}_{w(\mathbf{P}_{J,\ell}^+)}^{\mathbf{U}_\ell}(\cdot)$, $\text{S-Coind}_{w(\mathbf{P}_{J,\ell}^+)}^{\mathbf{U}_\ell}(\cdot)$ etc. are defined in a similar way.

3.3.3. **Lemma:** (see [Ar4])

(i) $\mathrm{Tor}^{w(\mathbf{B}^+_{\mathcal{A}})}_{\frac{\infty}{2}+k}(S^{\mathbf{U}^+_{\mathcal{A}}}_{\mathbf{U}_{\mathcal{A}}},\cdot)=0$ for $k\neq 0$;

(ii) $\mathrm{Ext}^{\frac{\infty}{2}+k}_{w(\mathbf{B}^+_{\mathcal{A}})}(S^{\mathbf{U}^-_{\mathcal{A}}}_{\mathbf{U}_{\mathcal{A}}},\cdot)=0$ for $k\neq 0$;

(iii) $\text{S-Ind}^{\mathbf{U}_{\mathcal{A}}}_{w(\mathbf{B}^+_{\mathcal{A}})}(\cdot)$ and $\text{S-Coind}^{\mathbf{U}_{\mathcal{A}}}_{w(\mathbf{B}^+_{\mathcal{A}})}(\cdot)$ define *exact* functors $w(\mathbf{B}^+_{\mathcal{A}})$-mod $\longrightarrow$ $\mathbf{U}_{\mathcal{A}}$-mod. □

Similar statements hold for the algebras $w(\mathbf{B}^+_\ell)$, $w(\mathbf{P}_{J,\mathcal{A}})$ and $w(\mathbf{P}_{J,\ell})$.

3.4. **Quasi-Verma modules.** We define the *quasi-Verma module* over the algebra $\mathbf{U}_{\mathcal{A}}$ (resp. $\mathbf{U}_\ell$) with the highest weight $w\cdot\lambda$ by

$$M^w_{\mathcal{A}}(w\cdot\lambda):=\text{S-Ind}^{\mathbf{U}_{\mathcal{A}}}_{w(\mathbf{B}^+_{\mathcal{A}})}(\mathcal{A}(\lambda))\ (\text{resp. } M^w_\ell(w\cdot\lambda):=\text{S-Ind}^{\mathbf{U}_\ell}_{w(\mathbf{B}^+_\ell)}(\mathbb{C}(\lambda))).$$

The *contragradient quasi-Verma module* $\mathbb{D}M^w_{\mathcal{A}}(w\cdot\lambda)$ (resp. $\mathbb{D}M^w_\ell(w\cdot\lambda)$) is defined by

$$\mathbb{D}M^w_{\mathcal{A}}(w\cdot\lambda):=\text{S-Coind}^{\mathbf{U}_{\mathcal{A}}}_{w(\mathbf{B}^+_{\mathcal{A}})}(\mathcal{A}(\lambda))\ (\text{resp. } \mathbb{D}M^w_\ell(w\cdot\lambda):=\text{S-Coind}^{\mathbf{U}_\ell}_{w(\mathbf{B}^+_\ell)}(\mathbb{C}(\lambda))).$$

We list the main properties of quasi-Verma modules.

3.4.1. **Proposition:** (see [Ar4])

(i) Fix a dominant integral weight $\lambda\in X$. Suppose that $\xi\in\mathbb{C}^*$ is not a root of unity. Then the $\mathbf{U}_\xi$-module $M^w_\xi(w\cdot\lambda):=M^w_{\mathcal{A}}(w\cdot\lambda)\otimes_{\mathcal{A}}\mathbb{C}$ (resp. $\mathbb{D}M^w_\xi(w\cdot\lambda):=\mathbb{D}M^w_{\mathcal{A}}(w\cdot\lambda)\otimes_{\mathcal{A}}\mathbb{C}$) is isomorphic to the usual Verma module $M_\xi(w\cdot\lambda)$ (resp. to the usual contragradient Verma module $\mathbb{D}M_\xi(w\cdot\lambda)$).

(ii) For any $\lambda\in X$ we have

$$\begin{aligned}\mathrm{ch}(M^w_{\mathcal{A}}(w\cdot\lambda))&=\mathrm{ch}(\mathbb{D}M^w_{\mathcal{A}}(w\cdot\lambda))=\mathrm{ch}(M^w_\ell(w\cdot\lambda))\\&=\mathrm{ch}(\mathbb{D}M^w_\ell(w\cdot\lambda))=\frac{e^{w\cdot\lambda}}{\prod_{\alpha\in R^+}(1-e^{-\alpha})}.\quad\square\end{aligned}$$

Thus for a dominant weight $\lambda$ one can consider $M^w_{\mathcal{A}}(w\cdot\lambda)$ as a flat family of modules over the quantum group for various values of the quantizing parameter with the fiber at the generic point equal to the Verma module $M_\xi(w\cdot\lambda)$.

**Remark:** Note that by definition $M^e_\ell(\lambda)=M_\ell(\lambda)$ and $\mathbb{D}M^e_\ell(\lambda)=\mathbb{D}M_\ell(\lambda)$, where $e$ denotes the unity element of the Weyl group. In particular for a dominant weight $\lambda$ we have a natural projection $M^e_\ell(\lambda)\longrightarrow W_\ell(\lambda)$ and a natural inclusion $\mathbb{D}W_\ell(\lambda)\hookrightarrow\mathbb{D}M^e_\ell(\lambda)$.

3.5. **The $\mathbf{U}_\ell(\mathfrak{sl}_2)$ case.** Let us investigate throughly quasi-Verma modules in the case of $\mathbf{U}_\ell=\mathbf{U}_\ell(\mathfrak{sl}_2)$. First we find the simple subquotient modules in the module $M^e_\ell(k\ell)=M_\ell(k\ell)$.

Recall the classification of the simple objects in the category of $X$-graded $\mathbf{U}^0_\ell$-semisimple $\mathbf{U}_\ell$-modules locally finite with respect to the action of $E_i$ and $E_i^{(\ell)}$, $i\in I$, obtained by Lusztig in [L2]. In the $\mathfrak{sl}_2$ case it looks as follows. Identify the weight lattice $X$ with $\mathbb{Z}$.

**Proposition:**

(i) For $0\leq k<\ell$ the simple $\mathfrak{u}_\ell(\mathfrak{sl}_2)$-module $L(k)$ is a restriction of a simple $\mathbf{U}_\ell(\mathfrak{sl}_2)$-module.

(ii) Any simple $\mathbf{U}_\ell(\mathfrak{sl}_2)$-module from the category described above is isomorphic to a module of the form $L(k)\otimes L(m\ell)$, where $0 \leq k < \ell$. Here the simple module $L(m\ell)$ is obtained from the simple $U(\mathfrak{sl}_2)$-module $L(m)$ via restricton using the map $\mathbf{U}_\ell(\mathfrak{sl}_2) \longrightarrow \mathbf{U}(\mathfrak{sl}_2)//\mathfrak{u}_\ell(\mathfrak{sl}_2) = U(\mathfrak{sl}_2)$. $\square$

Denote the only reflection in the Weyl group for $\mathfrak{sl}_2$ by $s$.

**Lemma:**

(i) $\operatorname{ch} M^e_\ell(0) = \operatorname{ch} L(0) + \operatorname{ch} L(-2) + \operatorname{ch} L(-2\ell)$.
(ii) For $k > 0$ we have $\operatorname{ch} M^e_\ell(k\ell) = \operatorname{ch} L(k\ell) + \operatorname{ch} L(k\ell - 2) + \operatorname{ch} L(-k\ell - 2) + \operatorname{ch} L(-(k+2)\ell)$.
(iii) $\operatorname{ch} M^s_\ell(s \cdot 0) = \operatorname{ch} L(-2) + \operatorname{ch} L(-2\ell)$.
(iv) For $k > 0$ we have $\operatorname{ch} M^s_\ell(s \cdot k\ell) = + \operatorname{ch} L(-k\ell - 2) + \operatorname{ch} L(-(k+2)\ell)$. $\square$

In fact it is easy to find the filtrations on quasi-Verma modules with simple subquotients that correspond to the character equalities above.

**Lemma:**

(i) For $k > 0$ there exist exact sequences

$$0 \longrightarrow L(k\ell - 2) \longrightarrow W(k\ell) \longrightarrow L(k\ell) \longrightarrow 0,$$
$$0 \longrightarrow L(-k\ell - 2) \longrightarrow M^s_\ell(s \cdot k\ell) \longrightarrow L(-(k+2)\ell) \longrightarrow 0.$$

(ii) There exists a filtration on $M^e_\ell(0)$ with subquotients as follows:

$$\operatorname{gr}^1 M^e_\ell(0) = L(0),\ \operatorname{gr}^2 M^e_\ell(0) = L(-2),\ \operatorname{gr}^3 M^e_\ell(0) = L(-2\ell).$$

(iii) For $k > 0$ there exists a filtration on $M^e_\ell(k\ell)$ with subquotients as follows:

$$\operatorname{gr}^1 M^e_\ell(k\ell) = L(k\ell),\ \operatorname{gr}^2 M^e_\ell(k\ell) = L(k\ell - 2),$$
$$\operatorname{gr}^3 M^e_\ell(k\ell) = L(-(k+2)\ell),\ \operatorname{gr}^4 M^e_\ell(k\ell) = L(-k\ell - 2). \quad \square$$

Thus we obtain the following statement.

3.5.1. **Proposition:** For any $k \geq 0$ there exists an exact sequence of $\mathbf{U}_\ell = \mathbf{U}_\ell(\mathfrak{sl}_2)$-modules

$$0 \longrightarrow M^s_\ell(s \cdot k\ell) \longrightarrow M^e_\ell(k\ell) \longrightarrow W_\ell(k\ell) \longrightarrow 0. \quad \square$$

A more accurate calculation of characters shows that the complex similar to the one from the previous Proposition exists also for non-$\ell$-divisible dominant highest weights.

**Proposition:** For any $\mu \geq 0$ there exists an exact sequence of $\mathbf{U}_\ell = \mathbf{U}_\ell(\mathfrak{sl}_2)$-modules

$$0 \longrightarrow M^s_\ell(s \cdot \mu) \longrightarrow M^e_\ell(\mu) \longrightarrow W_\ell(\mu) \longrightarrow 0. \quad \square$$

**Remark:**

(i) In fact it is easy to verify that for $\mu$ dominant the module $M^s_\ell(s\cdot\mu)$ is isomorphic to the *contragradient* Verma module $\mathbb{D}M_\ell(s \cdot \mu)$.
(ii) Note that if $\xi$ is not a root of unity then the usual BGG resolution in the $\mathfrak{sl}_2$ case provides an exact complex

$$0 \longrightarrow M^s_\xi(s \cdot k\ell) \longrightarrow M^e_\xi(k\ell) \longrightarrow L(k\ell) \longrightarrow 0.$$

Thus we see that the flat family of such complexes over $\mathbb{C}^* \setminus \{\text{roots of unity}\}$ is extended over the whole $\mathbb{C}^*$.

3.5.2. *The* $\mathbf{U}_{\mathcal{A}}(\mathfrak{s}l_2)$*-case.* In fact we will need the existence of *a complex of* $\mathbf{U}_{\mathcal{A}}(\mathfrak{s}l_2)$*-modules* similar to the one constructed in the previous subsection. Recall that by definition there exists a natural projection of $\mathbf{U}_{\mathcal{A}}$-modules $M_{\mathcal{A}}(\mu) \longrightarrow W_{\mathcal{A}}(\mu)$. Denote its kernel by $K$.

3.5.3. **Proposition:** The $\mathbf{U}_{\mathcal{A}}(\mathfrak{sl}_2)$-module $K$ is isomorphic to $\mathbb{D}M_{\mathcal{A}}(s\cdot\mu)$.
**Proof.** First note that $K$ has a highest weight vector $p$ of the weight $s\cdot\mu$. In particular there exists a natural map $K \longrightarrow \mathbb{D}M_{\mathcal{A}}(s\cdot\mu)$. Thus it is sufficient to check that the module $K$ is co-generated by this vector. This claim follows from the calculation below. Denote the highest weight vector in $M_{\mathcal{A}}(\mu)$ by $p_\mu$. Then evidently $p = F^{(\mu+1)}p_\mu$. Using the formulas from [L1], 6.4, for every $m > \mu$ we have

$$E^{(m-\mu-1)}\cdot F^{(m)}p_\mu = F^{(\mu+1)}\prod_{s=1}^{m-\mu-1}\frac{Kv^{-\mu-1-s+1}-K^{-1}v^{\mu+1+s-1}}{v^s-s^{-s}}v_\mu = -F^{(\mu+1)}p_\mu. \quad \square$$

**Corollary:** For every positive integer $\mu$ there exists an exact complex of $\mathbf{U}_{\mathcal{A}}(\mathfrak{sl}_2)$-modules

$$0 \longrightarrow M^s_{\mathcal{A}}(s\cdot\mu) \longrightarrow M^e_{\mathcal{A}}(\mu) \longrightarrow W_{\mathcal{A}}(\mu) \longrightarrow 0. \quad \square$$

We call the complex $M^s_{\mathcal{A}}(s\cdot\mu) \longrightarrow M^e_{\mathcal{A}}(\mu)$ the quasi-BGG complex for the weight $\mu$ and denote it by $B^\bullet_{\mathcal{A}}(\mu)$.

3.5.4. *The* $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{s}l_2)$*-case.* The crucial point in the proof of the exactness of the quasi-BGG complex for arbitrary root data $(Y, X, \dots)$ uses a geometric argument in positive characteristic. Thus the following Lemma is impor tant being the $\mathfrak{sl}_2$ case of the general picture.

**Lemma:**

(i) For each $i \in I$ the action of $K_i - 1$ on the complex of $\mathbf{U}_{\overline{\mathbb{F}}_\ell}$-modules $B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu) := B^\bullet_{\mathcal{A}}(\mu)\otimes_{\mathcal{A}}\overline{\mathbb{F}}_\ell$ is trivial, i. e. $B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu)$ becomes a complex of $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$-modules.

(ii) The complex $\mathbb{D}B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu)$ is isomorphic to the global Cousin complex $C^\bullet(\mu)$ for the line bundle $\mathcal{L}(\mu)$ on $\mathbf{P}^1_{\overline{\mathbb{F}}_\ell}$ discussed in 3.1. $\square$

3.6. **Construction of the quasi-BGG complex.** Here we extend the previous considerations to the case of the quantum group $\mathbf{U}_{\mathcal{A}}$ for arbitrary root data $(Y, X, \dots)$ of the finite type $(I, \cdot)$. Fix a dominant weight $\mu \in X$.

First we construct an inclusion $M^{w'}_{\mathcal{A}}(w'\cdot\mu) \hookrightarrow M^w_{\mathcal{A}}(w\cdot\mu)$ for a pair of elements $w', w \in W$ such that $\ell t(w') = \ell t(w) + 1$ and $w' > w$ in the Bruhat order on the Weyl group. In fact we can do it explicitly only for $w'$ and $w$ differing by a simple reflection: $w' = ws_i$, $i \in I$.

Consider the twisted quantum parabolic subalgebra $w(\mathbf{P}_{i,\mathcal{A}})$. Then $w(\mathbf{P}_{i,\mathcal{A}}) \supset w(\mathbf{B}^+_{\mathcal{A}})$ and $w(\mathbf{P}_{i,\mathcal{A}}) \supset ws_i(\mathbf{B}^+_{\mathcal{A}})$. Consider also the Levi quotient algebra $w(\mathbf{P}_{i,\mathcal{A}}) \longrightarrow w(\mathbf{L}_{i,\mathcal{A}})$. The algebra $w(\mathbf{L}_{i,\mathcal{A}})$ is isomorphic to $\mathbf{U}_{\mathcal{A}}(\mathfrak{s}l_2)\otimes_{\mathbf{U}^0_{\mathcal{A}}(\mathfrak{s}l_2)}\mathbf{U}^0_{\mathcal{A}}$.

By Proposition 3.5.1 we have a natural inclusion of $w(\mathbf{L}_{i,\mathcal{A}})$-modules

$$\text{S-Ind}^{w(\mathbf{L}_{i,\mathcal{A}})}_{ws_i(\mathbf{L}^+_{i,\mathcal{A}})}\mathcal{A}(\mu) \hookrightarrow \text{S-Ind}^{w(\mathbf{L}_{i,\mathcal{A}})}_{w(\mathbf{L}^+_{i,\mathcal{A}})}\mathcal{A}(\mu).$$

**Lemma:**

(i) $\text{S-Ind}^{\mathbf{U}_{\mathcal{A}}}_{w(\mathbf{B}^+_{\mathcal{A}})}(\cdot) = \text{S-Ind}^{\mathbf{U}_{\mathcal{A}}}_{w(\mathbf{P}_{i,\mathcal{A}})}\circ\text{S-Ind}^{w(\mathbf{P}_{i,\mathcal{A}})}_{w(\mathbf{B}^+_{\mathcal{A}})}(\cdot)$.

(ii) $\text{S-Ind}_{ws_i(\mathbf{B}_{\mathcal{A}}^+)}^{\mathbf{U}_{\mathcal{A}}}(\cdot) = \text{S-Ind}_{w(\mathbf{P}_{i,\mathcal{A}})}^{\mathbf{U}_{\mathcal{A}}} \circ \text{S-Ind}_{ws_i(\mathbf{B}_{\mathcal{A}}^+)}^{w(\mathbf{P}_{i,\mathcal{A}})}(\cdot)$.

(iii) $\text{S-Ind}_{w(\mathbf{B}_{\mathcal{A}}^+)}^{\mathbf{U}_{\mathcal{A}}}(\mathcal{A}(\mu)) = \text{S-Ind}_{w(\mathbf{P}_{i,\mathcal{A}})}^{\mathbf{U}_{\mathcal{A}}} \circ \text{Res}_{w(\mathbf{P}_{i,\mathcal{A}})}^{w(\mathbf{L}_{i,\mathcal{A}})} \circ \text{S-Ind}_{W(\mathbf{L}_{i,\mathcal{A}}^+)}^{w(\mathbf{L}_{i,\mathcal{A}})}(\mu)$.

(iv) $\text{S-Ind}_{ws_i(\mathbf{B}_{\mathcal{A}}^+)}^{\mathbf{U}_{\mathcal{A}}}(\mathcal{A}(\mu)) = \text{S-Ind}_{w(\mathbf{P}_{i,\mathcal{A}})}^{\mathbf{U}_{\mathcal{A}}} \circ \text{Res}_{w(\mathbf{P}_{i,\mathcal{A}})}^{w(\mathbf{L}_{i,\mathcal{A}})} \circ \text{S-Ind}_{ws_i(\mathbf{L}_{i,\mathcal{A}}^+)}^{w(\mathbf{L}_{i,\mathcal{A}})}(\mu)$. □

**Corollary:** For $w' = ws_i > w$ in the Bruhat order we have a natural inclusion of $\mathbf{U}_{\mathcal{A}}$-modules $i_{\mathcal{A}}^{ws_i,w}: M_{\mathcal{A}}^{ws_i}(ws_i \cdot \mu) \hookrightarrow M_{\mathcal{A}}^{w}(w \cdot \mu)$. □

Recall that if $v$ acts on $\mathbb{C}$ by $\xi$ that is not a root of unity then the $\mathbf{U}_\xi$-module $M_\xi^w(w\cdot\mu) := M_{\mathcal{A}}^w(w\cdot\mu)\otimes_{\mathcal{A}}\mathbb{C}$ is isomorphic to the usual Verma module $M_\xi(w\cdot\mu)$. Thus the morphism $i_\xi^{w',w}$ coincides with the standard inclusion of Verma modules constructed by J. Bernshtein, I.M. Gelfand and S.I. Gelfand in [BGG] that becomes a component of the differential in the BGG resolution. In other words we see that the flat family of inclusions $i_\xi^{ws_i,w}: M_\ell^{ws_i}(ws_i\cdot\mu) \hookrightarrow M_\xi^w(w\cdot\mu)$ defined for $\xi \in \mathbb{C}^*\setminus\{\text{roots of unity}\}$ can be extended naturally over the whole $\operatorname{Spec}\mathcal{A}$.

Iterating the inclusion maps we obtain a flat family of submodules $i_\xi^w(M_\xi^w(w\cdot\mu)) \subset M_\xi^e(\mu)$ for $\xi \in \operatorname{Spec}\mathcal{A}$, $w \in W$, providing an extension of the standard lattice of Verma submodules in $M_\xi(\mu)$ defined a priori for $\xi \in \mathbb{C}^* \setminus \{\text{roots of unity}\}$.

3.6.1. **Lemma:** For a pair of elements $w', w \in W$ such that $\ell t(w') = \ell t(w) + 1$ and $w' > w$ in the Bruhat order we have

$$i_{\mathcal{A}}^{w'}(M_{\mathcal{A}}^{w'}(w'\cdot\mu)) \hookrightarrow i_{\mathcal{A}}^{w}(M_{\mathcal{A}}^{w}(w\cdot\mu)).$$

**Proof.** To prove the statement note that the condition $\{A_\xi$ is a submodule in $B_\xi\}$ is a closed condition in a flat family. □

Now using the standard combinatorics of the classical BGG resolution we obtain the following statement.

3.6.2. **Theorem:** There exists a complex of $\mathbf{U}_{\mathcal{A}}$-modules $B_C^\bullet A(\mu)$ with

$$B_{\mathcal{A}}^{-k}(\mu) = \bigoplus_{w\in W, \ell t(w)=k} M_{\mathcal{A}}^w(w\cdot\mu)$$

and with differentials provided by direct sums of the inclusions $i_{\mathcal{A}}^{w',w}$. □

3.6.3. **Definition:** We call the complex $B_{\mathcal{A}}^\bullet(\mu)$ the *quasi-BGG complex* for the dominant weight $\mu \in X$.

Denote the complex of $\mathfrak{u}_\ell$-modules $B_{\mathcal{A}}(\mu)\otimes_{\mathcal{A}}\mathbb{C}$ by $B_\ell^\bullet(\mu)$. Below we prove that for $\ell$ prime the quasi-BGG complex $B_\ell^\bullet(\mu)$ is in fact quasiisomorphic to the Weyl module $W_\ell(\mu)$.

3.7. **Quasi-BGG complexes in positive characteristic.** Note that we can perform specialization of the quasi-BGG complex over $\mathcal{A}$ into a root of unity in two steps. Consider the complex of $\mathbf{U}_{\mathcal{A}'_\ell}$-modules $\mathbb{D}B_{\mathcal{A}'_\ell}^\bullet(\mu) := \mathbb{D}B_{\mathcal{A}}^\bullet(\mu)\otimes_{\mathcal{A}}\mathcal{A}'_\ell$. Evidently its specialization into the generic point of $\operatorname{Spec}\mathcal{A}'_\ell$ coincides with $\mathbb{D}B_\ell^\bullet(\mu)$. On the other hand consider the specialization of the complex into the spec ial point $\operatorname{Spec}\overline{\mathbb{F}}_\ell \hookrightarrow \operatorname{Spec}\mathcal{A}'_\ell$. The following statement is similar to Lemma 3.5.4.

3.7.1. **Proposition:**

(i) For each $i \in I$ the action of $K_i - 1$ on the complex of $\mathbf{U}_{\overline{\mathbb{F}}_\ell}$-modules $B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu) := B^\bullet_{\mathcal{A}'_\ell}(\mu) \otimes_{\mathcal{A}'_\ell} \overline{\mathbb{F}}_\ell$ is trivial, i. e. $B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu)$ becomes a complex of $U_{\overline{\mathbb{F}}_\ell}(\mathfrak{g})$-modules.
(ii) The complex is isomorphic to the global Cousin complex $C^\bullet(\mu)$ for the line bundle $\mathcal{L}(\mu)$ on $\mathcal{B}_{\overline{\mathbb{F}}_\ell}$ discussed in 3.1. □

**Corollary:** The complex $\mathbb{D}B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu)$ is quasiisomorphic to the contragradient Weyl module $\mathbb{D}W_{\overline{\mathbb{F}}_\ell}(\mu) = \mathbb{D}W_{\mathcal{A}}(\mu) \otimes_{\mathcal{A}} \overline{\mathbb{F}}_\ell = H^0(\mathcal{B}_{\overline{\mathbb{F}}_\ell}, \mathcal{L}(\mu))$. □
Surprisingly this result proves the exactness of the quasi-BGG complex over $\mathbf{U}_\ell$.
**Theorem:** For $\ell$ prime the complex $\mathbb{D}B^\bullet_{\overline{\mathbb{F}}_\ell}(\mu)$ is quasiisomorphic to the contragradient Weyl module $\mathbb{D}W_\ell(\mu)$.
**Proof.** Denote the evident morphism $\mathbb{D}W_{\mathcal{A}'_\ell}(\mu) \longrightarrow \mathbb{D}B^\bullet_{\mathcal{A}'_\ell}(\mu)$ by $can_{\mathcal{A}'_\ell}$. Consider the complex Cone $\left(can_{\mathcal{A}'_\ell}\right)$. By the previous Corollary the specialization of the complex into the special point $\operatorname{Spec} \overline{\mathbb{F}}_\ell \hookrightarrow \operatorname{Spec} \mathcal{A}'_\ell$ is exact. By the Nakayama lemma it follows that the specialization of the complex into the generic point of $\operatorname{Spec} \mathcal{A}'_\ell$ is also exact. In particular the complex $\operatorname{Cone}(can_\ell) := \operatorname{Cone}\left(can_{\mathcal{A}'_\ell}\right) \otimes_{\mathcal{A}'_\ell} \mathbb{C}$, where $v$ acts on $\mathbb{C}$ by the $\ell$-th root of unity, is exact as well. □

3.8. **Semiinfinite cohomology with coefficients in quasi-Verma modules.** Recall the following construction that plays crucial role in considerations of Ginzburg and Kumar in [GK].

Let $\left(\mathbf{B}^+_\ell\text{-mod}\right)^{\text{fin}}$ (resp. $(\mathbf{U}_\ell\text{-mod})^{\text{fin}}$, resp. $(U(\mathfrak{b}^+)\text{-mod})^{\text{fin}}$, resp. $(U(\mathfrak{g})\text{-mod})^{\text{fin}}$) be the category of finite dimensional $X$-graded modules over the corresponding algebra with the action of the Cartan subalgebra semisimple and well defined with respect to the $X$-gradings. Consider the functors:

$$\operatorname{Coind}_{\mathbf{B}^+_\ell}^{\mathbf{U}_\ell} : \mathbf{B}^+_\ell\text{-mod} \longrightarrow \mathbf{U}_\ell\text{-mod}; \ \left(\operatorname{Coind}_{\mathbf{B}^+_\ell}^{\mathbf{U}_\ell}\right)^{\text{fin}} : \left(\mathbf{B}^+_\ell\text{-mod}\right)^{\text{fin}} \longrightarrow (\mathbf{U}_\ell\text{-mod})^{\text{fin}};$$
$$\operatorname{Coind}_{U(\mathfrak{b}^+)}^{U(\mathfrak{g})} : U(\mathfrak{b}^+)\text{-mod} \longrightarrow U(\mathfrak{g})\text{-mod};$$
$$\left(\operatorname{Coind}_{U(\mathfrak{b}^+)}^{U(\mathfrak{g})}\right)^{\text{fin}} : \left(U(\mathfrak{b}^+)\text{-mod}\right)^{\text{fin}} \longrightarrow (U(\mathfrak{g})\text{-mod})^{\text{fin}},$$
$$(\cdot)^{\mathfrak{b}^+_\ell} : \mathbf{B}^+_\ell\text{-mod} \longrightarrow U(\mathfrak{b}^+)\text{-mod} \text{ and } \left(\mathbf{B}^+_\ell\text{-mod}\right)^{\text{fin}} \longrightarrow \left(U(\mathfrak{b}^+)\text{-mod}\right)^{\text{fin}};$$
$$(\cdot)^{\mathfrak{u}_\ell} : \mathbf{U}_\ell\text{-mod} \longrightarrow U(\mathfrak{g})\text{-mod} \text{ and } (\mathbf{U}_\ell\text{-mod})^{\text{fin}} \longrightarrow (U(\mathfrak{g})\text{-mod})^{\text{fin}},$$

where $(\cdot)^{\text{fin}}$ denotes taking the maximal finite dimensional submodule in $(\cdot)$ and $(\cdot)^{\mathfrak{b}^+_\ell}$ (resp. $(\cdot)^{\mathfrak{u}_\ell}$) denotes taking $\mathfrak{b}^+_\ell$- (resp. $\mathfrak{u}_\ell$)-invariants.
**Proposition:** (see [GK])

(i) $(\cdot)^{\mathfrak{u}_\ell} \circ \operatorname{Coind}_{\mathbf{B}^+_\ell}^{\mathbf{U}_\ell} = \operatorname{Coind}_{U(\mathfrak{b}^+)}^{U(\mathfrak{g})} = (\cdot)^{\mathfrak{b}^+_\ell}$;
(ii) $(\cdot)^{\mathfrak{u}_\ell} \circ \left(\operatorname{Coind}_{\mathbf{B}^+_\ell}^{\mathbf{U}_\ell}\right)^{\text{fin}} = \left(\operatorname{Coind}_{U(\mathfrak{b}^+)}^{U(\mathfrak{g})}\right)^{\text{fin}} (\cdot)^{\mathfrak{b}^+_\ell}$. □

The semiinfinite analogue for the first part of the previous statement looks as follows. Fix $w \in W$. Consider the functors:

$$\begin{gathered}\text{S-Coind}_{w(\mathbf{B}_\ell^+)}^{\mathbf{U}_\ell}:\ \mathsf{D}(w(\mathbf{B}_\ell^+)\text{-mod}) \longrightarrow \mathsf{D}(\mathbf{U}_\ell\text{-mod}),\\ \text{S-Coind}_{U(w(\mathfrak{b}^+))}^{U(\mathfrak{g})}:\ \mathsf{D}(U(w(\mathfrak{b}^+))\text{-mod}) \longrightarrow \mathsf{D}(U(\mathfrak{g})\text{-mod}),\\ \mathrm{Ext}_{w(\mathfrak{b}_\ell^+)}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\cdot):\ \mathsf{D}(w(\mathbf{B}_\ell^+)\text{-mod}) \longrightarrow \mathsf{D}(U(w(\mathfrak{b}^+))\text{-mod}),\\ \mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\cdot):\ \mathsf{D}(w(\mathbf{U}_\ell)\text{-mod}) \longrightarrow \mathsf{D}(U(\mathfrak{g}))\text{-mod}).\end{gathered}$$

3.8.1. **Theorem:** We have

$$\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\cdot)\circ \text{S-Coind}_{w(\mathbf{B}_\ell^+)}^{\mathbf{U}_\ell} = \text{S-Coind}_{U(w(\mathfrak{b}^+))}^{U(\mathfrak{g})}\circ \mathrm{Ext}_{w(\mathfrak{b}_\ell^+)}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\cdot).$$

**Proof.** To simplify the notations we work with semiinfinite homology and semiinfinite induction instead of semiinfinite cohomology and semiinfinite coinduction. By [Ar1], Appendix B, every convex complex of $w(\mathbf{B}_\ell^+)$-modules is quasiisomorphic to a K-semijective convex complex. Consider a K-semijective convex complex of $w(\mathbf{B}_\ell^+)$-modules $SS^\bullet$. Note that both semiinfinite induction functors are exact and take K-semijective complexes to K-semijective complexes. Thus we have

$$\begin{aligned}\mathrm{Tor}^{\mathfrak{u}_\ell}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}},\cdot)\circ \text{S-Ind}_{w(\mathbf{B}_\ell^+)}^{\mathbf{U}_\ell}(SS^\bullet) &= \mathrm{Tor}^{\mathfrak{u}_\ell}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \mathrm{Tor}^{w(\mathbf{B}_\ell^+)}_{\frac{\infty}{2}+\bullet}(S^{\mathbf{U}_\ell^+}_{\mathbf{U}_\ell}, SS^\bullet))\\ = \mathrm{Tor}^{w(\mathbf{B}_\ell^+)}_{\frac{\infty}{2}+\bullet}(\mathrm{Tor}^{\mathfrak{u}_\ell}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, S^{\mathbf{U}_\ell^+}_{\mathbf{U}_\ell}), SS^\bullet)) &= \mathrm{Tor}^{w(\mathbf{B}_\ell^+)}_{\frac{\infty}{2}+\bullet}(S^{U(\mathfrak{n}^+)}_{U(\mathfrak{g})}, SS^\bullet)\\ = \mathrm{Tor}^{w(\mathbf{B}_\ell^+)}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, S^{U(\mathfrak{n}^+)}_{U(\mathfrak{g})})\otimes SS^\bullet) &= \mathrm{Tor}^{U(w(\mathfrak{b}^+))}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \mathrm{Tor}^{w(\mathfrak{b}_\ell^+)}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, S^{U(\mathfrak{n}^+)}_{U(\mathfrak{g})})\otimes SS^\bullet))\\ = \mathrm{Tor}^{U(w(\mathfrak{b}^+))}_{\frac{\infty}{2}+\bullet}(S^{U(\mathfrak{n}^+)}_{U(\mathfrak{g})}), \mathrm{Tor}^{w(\mathfrak{b}_\ell^+)}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, SS^\bullet)) &= \text{S-Ind}^{U(\mathfrak{g})}_{U(w(\mathfrak{b}^+))}\circ \mathrm{Tor}^{w(\mathfrak{b}_\ell^+)}_{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, SS^\bullet).\end{aligned}$$

Here we used the fact that the subalgebra $w(\mathfrak{b}_\ell^+) \subset w(\mathbf{B}_\ell^+)$ is normal with the quotient algebra equal to $U(w(\mathfrak{b}^+))$. □

**Corollary:**

$$\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \mathbb{D}M_\ell^w(w\cdot \ell\lambda)) = H^{\sharp(R^+)}_{T^*_{S_w}(\mathcal{B})}(\widetilde{\mathcal{N}}, \pi^*\mathcal{L}(\lambda))$$

as a module over both $U(\mathfrak{g})$ and $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}) = H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$. □

Recall that for the Springer-Grothendieck resolution of the nilpotent cone $\mu : \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}$ we have $\mu^{-1}(\mathfrak{n}^+) = \bigsqcup_{w\in W} T^*_{S_w}(\mathcal{B})$.

**Proposition:**

(i) There exists a filtration on $H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \pi^*\mathcal{L}(\lambda))$ with the subquotients equal to $H^{\sharp(R^+)}_{T^*_{S_w}(\mathcal{B})}(\widetilde{\mathcal{N}}, \pi^*\mathcal{L}(\lambda))$, for $w \in W$.

(ii) $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \mathbb{D}B_\ell^\bullet(\ell\lambda)) = H^{\sharp(R^+)}_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, \pi^*\mathcal{L}(\lambda))$. □

Comparing this statement with Theorem 3.7.1 we obtain the main result of the section.

3.8.2. **Theorem:** For $\ell$ prime $\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \mathbb{D}W(\ell\lambda)) = H^0_{\mu^{-1}(\mathfrak{n}^+)}(\widetilde{\mathcal{N}}, p^*\mathcal{L}(\lambda))$ both as a $\mathbf{U}(\mathfrak{g})$-module and as a $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})$-module. □

## 4. Further results and conjectures.

In this section we present several facts without proof. We also formulate some conjectures concerning possible origin of the quasi-BGG complex.

4.1. **Alternative triangular decompositions of $\mathfrak{u}_\ell$.** Note that the definition of semiinfinite cohomology starts with specifying a triangular decomposition of a *graded* algebra $\mathfrak{u}$. Fix a subset $J \subset I$. Instead of the usual height function consider the linear map $\mathrm{ht}_J: \ X \longrightarrow \mathbb{Z}$ defined on the elements $i', i \in I$ by $\mathrm{ht}_J(i') = 0$ for $i \in J$ and $\mathrm{ht}_J(i') = 1$ otherwize and extended to the whole $X$ by linearity. Now we work in the category of complexes of $X$-graded $\mathfrak{u}_\ell$-modules satisfying conditions of concavity and convexity with respect to the $\mathbb{Z}$-grading obtained from the $X$-grading with the help of the function $\mathrm{ht}_J$.

Consider the triangular decomposition of the small quantum group $\mathfrak{u}_\ell = \mathfrak{p}_{J,\ell}^- \otimes \mathfrak{u}_{J,\ell}^+$, where $\mathfrak{p}_{J,\ell}^-$ denotes the small quantum negative parabolic subalgebra in $\mathfrak{u}_\ell$ corresponding to the subset $J \subset I$ and $\mathfrak{u}_{J,\ell}^+$ denotes the quantum analogue of the nilpotent radical in $\mathfrak{p}_J^+$ defined with the help of Lusztig generators of $\mathbf{U}_\ell$ and $\mathfrak{u}_\ell$ (see [L1]).

Then it is known that the subalgebra $\mathfrak{u}_{J,\ell}^+$ in $\mathfrak{u}_\ell$ is Frobenius just like $\mathfrak{u}_\ell^+$. Thus it is possible to use the general definition of semiinfinite cohomology presented in 2.3. Denote the corresponding functor by $\mathrm{Ext}_{\mathfrak{u}_\ell, J}^{\frac{\infty}{2}+\bullet}(*, *)$.

On the other hand consider the classical negative parabolic subalgebra $\mathfrak{p}_J^- \subset \mathfrak{g}$ and its nillradical $\mathfrak{n}_J^-$. Choose the standard $X$-homogeneous root basis $\{f_\alpha\}$ in the space $\mathfrak{n}_J^-$. Consider the subset in $\mathcal{N}^{(J)} \subset \mathcal{N}$ annihilated by all the elements of the base dual to $\{f_\alpha\}$.

4.1.1. **Theorem:** $\mathrm{Ext}_{\mathfrak{u}_\ell, J}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \widetilde{\longrightarrow} H_{\mathcal{N}^{(J)}}^{\dim(\mathfrak{n}_J^-)}(\mathcal{N}, \mathcal{O}_\mathcal{N})$ as $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-modules. $\square$

4.2. **Contragradient Weyl modules with non-$\ell$-divisible highest weights.** Fix a *dominant* weight of the form $\ell\lambda + w \cdot 0$, where $\lambda \in X$ and $w \in W$. It is known that all the dominant weights in the linkage class containing 0 look like this. Consider the contragradient Weyl module $\mathbb{D}W(\ell\lambda + w \cdot 0)$. The following statement generalizes Corollary 2.5.3. Its proof is similar to the proof of Conjecture 2.5.2.

4.2.1. **Theorem:**

$$\mathrm{ch}\left(\mathrm{Ext}_{\mathfrak{u}_\ell}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \mathbb{D}W(\ell\lambda + w \cdot 0)), t\right)$$

$$= \frac{t^{-\sharp(R^+)+l(w)}}{\prod_{\alpha \in R^+}(1 - e^{-\ell\alpha})} \sum_{v \in W} \frac{e^{v(\ell\lambda)} t^{2l(v)}}{\prod_{\alpha \in R^+, v(\alpha) \in R^+}(1 - t^2 e^{-\ell\alpha}) \prod_{\alpha \in R^+, v(\alpha) \in R^-}(1 - t^{-2} e^{-\ell\alpha})}. \quad \square$$

4.3. **Contragradient Weyl modules: alternative triangular decompositions.** Fix the triangular decomposition of the small quantum group $\mathfrak{u}_\ell$ like in 4.1. A natural generalization of Conjecture 2.5.2 to the case of the parabolic triangular decomposition looks as follows. We keep the notations from 4.1.

4.3.1. **Conjecture:** $\mathrm{Ext}_{\mathfrak{u}_\ell, J}^{\frac{\infty}{2}+\bullet}(\underline{\mathbb{C}}, \mathbb{D}W(\ell\lambda)) \widetilde{\longrightarrow} H_{\mathcal{N}^{(J)}}^{\dim(\mathfrak{n}_J^-)}(\mathcal{N}, \mu_* p^* \mathcal{L}(\lambda))$ as a module over $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$. $\square$

4.4. **Connection with affine Kac-Moody algebras.** Finally we would like to say a few words about a possible explanation for the existence of quasi-Verma modules and quasi-BGG resolutions.

Suppose for simplicity that the root data $(Y, X, \dots)$ are *simply laced*, i. e. the corresponding Cartan matrix is symmetric. Consider the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ corresponding to $\mathfrak{g}$. Fix a *negative* level $-h^\vee + k$, where $k \in 1/2\mathbb{Z}_{<0} \setminus \mathbb{Z}_{<0}$ and $h^\vee$ denotes the dual Coxeter number for chosen root data of the finite type. Consider the Kazhdan-Lusztig category $\widetilde{\mathcal{O}}_{-k}$ of $\mathfrak{g} \otimes \mathbb{C}[t]$-integrable finitely generated $\hat{\mathfrak{g}}$-modules diagonizible with respect to the Cartan subalgebra in $\hat{\mathfrak{g}}$ at the level $-2h^\vee - k$. Kazhdan and Lusztig showed that the category $\widetilde{\mathcal{O}}_k$ posasses a structure of a rigid tensor category with the *fusion* tensor product $\dot{\otimes}$. Moreover, they proved the following statement.

4.4.1. **Theorem:** (see [KL 1,2,3,4]) Let $\ell = -2k$. Then the tensor category $(\widetilde{\mathcal{O}}_k, \dot{\otimes})$ is equivalent to the category $(\mathbf{U}_\ell\text{-mod})^{\text{fin}}$ with the tensor product provided by the Hopf algebra structure on $\mathbf{U}_\ell$. □

We denote the functor $(\widetilde{\mathcal{O}}_k, \dot{\otimes}) \longrightarrow \left((\mathbf{U}_\ell\text{-mod})^{\text{fin}}, \otimes\right)$ providing the equivalence of categories by $\widetilde{\mathsf{kl}}$. Consider the usual category $\mathcal{O}_k$ for $\hat{\mathfrak{g}}$ at the same level. Finkelberg constructed a functor $\mathsf{kl}: \ \mathcal{O}_k \longrightarrow \mathbf{U}_\ell\text{-mod}$ extending the functor $\widetilde{\mathsf{kl}}$ (see [F]). Note that the functor $\mathsf{kl}$ has no chance to be an equivalence of categories because it is known not to be exact.

Fix a dominant (resp. arbitrary) weight $\lambda \in X$. Consider now the contragradient Weyl module $\mathbb{D}\mathcal{W}(\lambda) = \text{Coind}_{U(\mathfrak{g}\otimes\mathbb{C}[t])}^{U_{-2h^\vee-k}(\hat{\mathfrak{g}})} L(\lambda)$ and the contragradient Verma module $\mathbb{D}\mathcal{M}(\lambda) = \text{Coind}_{U(\mathfrak{g}\otimes\mathbb{C}[t])}^{U_{-2h^\vee-k}(\hat{\mathfrak{g}})} \mathbb{D}M(\lambda)$ over $\hat{\mathfrak{g}}$ at the level $-2h^\vee - k$ , where $L(\lambda)$ (resp. $M(\lambda)$) denotes the simple module (resp. the contragradient Verma module over $\mathfrak{g}$ with the highest weight $\lambda$. Then the usual contragradient BGG resolution of $L(\lambda)$ provides a resolution $\mathbb{D}\mathbf{B}(\lambda)$ of the contragradient Weyl module $\mathbb{D}\mathcal{W}(\lambda)$ consisting of direst sums of contragradient Verma modules of the form $\mathbb{D}\mathcal{M}(w \cdot \lambda)$, where $\in W$. It is known that the Kazhdan-Lusztig functor takes Weyl and contragradient Weyl modules over $\hat{\mathfrak{g}}$ to Weyl (resp. contragradient Weyl) modules over $\mathbf{U}_\ell$.

4.4.2. **Conjecture:** The functor $\mathsf{kl}$ takes $\mathbb{D}\mathcal{M}(w \cdot \ell\lambda)$ to the contragradient quasi-Verma module $\mathbb{D}M_\ell^w(w \cdot \ell\lambda)$. Moreover the complex $\mathsf{kl}(\mathbb{D}\mathbf{B}(\ell\lambda))$ is quasiisomorphic to $\mathbb{D}W(\ell\lambda)$. □

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